On the Combining of Correlated Random Measures with Application to Graph-Based Receivers

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Abstract—Nowadays, message combining is an essential component in most digital communication systems. Correlation between random measures has a significant impact on the combining process. In order to provide the best estimate after combining, correlation must be considered. In many applications correlation is obvious, e.g. correlation in the time, frequency, and/or spatial domain of a radio channel. In other cases, correlation is more concealed. In this paper, two methods to combine correlated random values are presented and applied to a graph-based iterative receiver. It is explained why correlation in the message exchange arises and how it can be taken into account in the message combining step. Simulation results are provided showing the performance gains when correlation is considered.

I. INTRODUCTION

In modern communication systems, an increasing amount of information is generated due to the implementation of sensor networks and/or multiple transmit antennas. Random measures at the receiver side are not necessarily uncorrelated and have to be combined appropriately. Ignoring the correlation results in biased estimates and may degrade the overall system performance, while it typically reduces the computational complexity. It is well known that log-likelihood ratios (LLRs) of a single random variable are additive if the observations are independent [1]. However, if the observations are correlated, the LLRs should be weighted before superposition. For multivariate Gaussian random variables, the weighting depends on the elements of the corresponding covariance matrix. It is reported in [2] that exploiting the correlation between intrinsic and extrinsic LLR values improves the performance of turbo processing when employing suboptimum constituent decoders. Furthermore, it is shown in [3] that for Rician channels the performance of maximum ratio combining (MRC) is greatly improved by taking the correlation between receive antennas into account. Despite a few exceptions like [2, 3], the knowledge of how to combine correlated variables is not widely spread in the communications community.

Message passing algorithms in combination with factor graphs offer promising solutions for joint channel estimation and data detection while maintaining a low complexity at a desirable BER/MSE performance. The flexibility of the framework allows the utilization of a variety of detection and estimation algorithms. In an attempt to reduce the computational complexity, suboptimum detection algorithms as well as suboptimum graph structures are often applied [4, 5]. Due to these strategies, correlation between exchanged messages is inherently introduced, which can significantly degrade the performance of the sum-product algorithm if not taken care of. So far, none of the published graph-based receivers does consider the correlation in the message combining step.

In this paper, two cases of combining correlated random measures are studied. First, combining multiple observations depending on a single variable (modeled as \( p(y_1, \ldots, y_N|x) \)) and, second, combining multiple variables given a single observation (modeled as \( p(y|x_1, \ldots, x_N) \)). Exemplary, the presented methods to combine correlated random measures are applied to the graph-based iterative receiver introduced in [5]. Monte Carlo simulations are provided, which illustrate the performance gain when correlation information is considered for the message combining.

Throughout the paper, we adopt the following notation conventions: Bold-face capital and lower-case letters stand for matrices and vectors of appropriate dimensions, respectively. \( \Sigma_{ij} \) refers to the \( i \)th row and \( j \)th column of a matrix \( \Sigma \) and \( i_N \) denotes a unit vector of length \( N \). Furthermore, \(( \cdot )^{-1}\) represent the matrix inverse and \(( \cdot )^T\) the transpose operator.

II. COMBINATION OF CORRELATED RANDOM MEASURES

Generally, correlation between multiple Gaussian distributed random variables can be well represented by a multivariate Gaussian distribution. In most digital communication systems, however, two extreme cases can be identified: (1) multiple observations are combined to represent a single variable \( p(y_1, \ldots, y_N|x) \) and (2) multiple variables are combined, which are instances of the same observation \( p(y|x_1, \ldots, x_N) \). The first case appears in numerous applications, such as sensor networks and/or systems with one transmit antenna and multiple receive antennas providing multiple observations of, for example, a transmitted signal. The latter case is typically observed when a phenomenon is represented by multiple variables, for instance, symbols transmitted by multiple antennas are utilized to estimate one channel coefficient.

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While correlation between different random measures is obvious in some cases, it might be inadvertently introduced in other applications. For example, when considering multiple transmit and/or receive antennas, spatial correlation is often considered. Concerning channel estimation, correlation in time and/or frequency domain is obvious as well. However, the inherent correlation between intrinsic and extrinsic information in suboptimum detectors is less understood. In the following, the combination of correlated observations as well as the combination of correlated variables is studied in Section II-A and Section II-B, respectively.

A. Combination of Multiple Correlated Observations

In digital communications, multiple observations can efficiently be combined by utilizing log-likelihood values (L-values). Accordingly, the combination of correlated L-values is investigated in the following. The extrinsic log-likelihood ratio is defined as follows:

\[ L(y|x) = \log_{e} \frac{p_{y|x}(y|x = +1)}{p_{y|x}(y|x = -1)}. \]  

(1)

The conditional probability density function (pdf) \( p(y|x) \) is defined as

\[ p(y|x) = \frac{1}{(2\pi)^{N/2}\Sigma^{1/2}} \exp \left( -\frac{1}{2} (y - \mu_{x})^{T} \Sigma^{-1} (y - \mu_{x}) \right), \]  

(2)

where \( \Sigma \) refers to the covariance matrix, \( y = [y_{1}, \ldots, y_{N}]^{T} \) is the received data sequence, and \( x \) denotes the transmitted data symbol [1]. If the transmitted data symbol experiences uncorrelated errors, the covariance matrix \( \Sigma \) has only non-zero entries along its main diagonal. Thus, the calculation of (1) can be simplified to the superposition of the individual L-values. The reliability information of all received symbols can be combined according to

\[ L(y|x) = \sum_{i=1}^{N} L(y_{i}|x), \]  

(3)

where \( L(y_{i}|x = \pm 1) = \frac{2}{\sigma^{2}_{i}} y_{i} \) and \( \sigma^{2}_{i} \) is the corresponding entry within the covariance matrix. However, this is only true for uncorrelated observations. Given correlated L-values \( L(y_{i}|x) \), equal gain combining according to (3) is too optimistic, i.e., the magnitude of the resulting LLR is too large [2]. This is easily seen, when two L-values from completely correlated observations \( L(y_{1}) = L(y_{2}) \) are assumed and combined. The true L-value is \( L = L(y_{1}) = L(y_{2}) \) since no new information is introduced by the second value. Given (3), the result would be \( L = 2 \cdot L(y_{1}) \), which is clearly wrong. The problem can be solved by a weighted superposition of L-values. The correct weighting of the L-values can be calculated as follows:

\[ L(y|x) = \sum_{i=1}^{N} \alpha_{i} L(y_{i}|x). \]  

(4)

Inserting (2) into (1) yields

\[ L(y|x) = \frac{1}{2} \left[ -(y - 1)^{T} \Sigma^{-1} (y - 1) + (y + 1)^{T} \Sigma^{-1} (y + 1) \right] \]

\[ = \sum_{i=1}^{N} y_{i} \left[ 2 \Sigma^{-1} + \sum_{j=1, j \neq i}^{N} (\Sigma^{-1} + \Sigma^{-1}) \right] \]

\[ = \sum_{i=1}^{N} 2 \lambda_{i} \left[ \Sigma^{-1} + \sum_{j=1, j \neq i}^{N} \Sigma^{-1} \right] \]

\[ = \sum_{i=1}^{N} L(y_{i}|x) \cdot \frac{\sigma^{2}_{i} \lambda_{i}}{a_{i}}. \]  

(5)

Thus, the overall log-likelihood ratio is a weighted sum of the individual observations. For uncorrelated observations, i.e. \( \rho = 0, a_{i} = 1 \) and (5) is equivalent to (3). Furthermore, no improvement is achieved when \( \rho \sigma_{1} = \sigma_{2} \) and/or \( \rho \sigma_{2} = \sigma_{1} \).

B. Combination of Multiple Correlated Variables

In factor graphs the true pdf of a random variable is often approximated by a Gaussian pdf in order to reduce complexity. Multiple Gaussian pdfs, which represent individual estimates of a random variable, are combined within the factor graph to increase the estimation accuracy. It is a common assumption that these individual pdfs are uncorrelated. The best linear unbiased estimator (BLUE) combines individual uncorrelated variables \( x_{i} \sim \mathcal{N}(\mu_{i}, \sigma^{2}_{i}) \) as follows:

\[ \hat{\mu} = \frac{\sum_{i=1}^{N} \frac{\mu_{i}}{\sigma^{2}_{i}}}{\sum_{i=1}^{N} \frac{1}{\sigma^{2}_{i}}}, \quad \sigma^{2} = \frac{1}{\sum_{i=1}^{N} \frac{1}{\sigma^{2}_{i}}}. \]  

(6)

If the individual messages are correlated, the message combining process has to be changed accordingly. Hence, the combination of correlated random variables is investigated in more detail in the following.

In order to obtain the lowest possible mean squared error of an estimate \( \hat{\mu} \) in terms of BLUE, it is required that

1) \( \hat{\mu} \) is a linear combination of individual estimates \( \hat{\mu}_{i} \);
2) \( \hat{\mu} \) provides an unbiased estimate of the true value \( \mu \);
3) \( \hat{\mu} \) has the lowest possible variance.

A general way to express Condition 1 is given by

\[ \hat{\mu} = \sum_{i=1}^{N} \alpha_{i} \hat{\mu}_{i}. \]  

(7)

Condition 2 requires \( \hat{\mu} \) to be unbiased, i.e. \( \text{E} \{ \hat{\mu} \} = \mu \). Assuming that the individual estimates \( \hat{\mu}_{i} \) are unbiased, a normalization constraint is required such that Condition 2 is fulfilled:

\[ \sum_{i=1}^{N} \alpha_{i} = 1. \]  

(8)
The variance of the estimated parameter is given by
\[ \sigma^2 = \alpha^T \Sigma \alpha, \] (10)
where \( \alpha \) is the column vector of the weighting factors \( \alpha_i \).

The new mean value of the combined Gaussian function according to (7) is calculated as
\[ \hat{\mu} = \alpha_1 \cdot \mu_1 + \alpha_2 \cdot \mu_2. \] (12)

Accordingly, the weighting factors are
\[ \alpha_1 = \frac{\sigma_2 (\sigma_2 - \rho \sigma_1)}{\sigma_1^2 - 2\rho \sigma_1 \sigma_2 + \sigma_2^2}, \] (13)
\[ \alpha_2 = \frac{\sigma_1 (\sigma_1 - \rho \sigma_2)}{\sigma_1^2 - 2\rho \sigma_1 \sigma_2 + \sigma_2^2}. \] (14)

The variance of the estimated parameter is given by
\[ \sigma^2 = (1 - \rho^2) \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2 + 2\rho \sigma_1 \sigma_2 + \sigma_2^2}. \] (15)

It is obvious that (6) and (15) are equivalent if \( \rho = 0 \). Furthermore, the similarity of (4) and (7) is evident. Considering the fact that the received observations \( y_1 \) and \( y_2 \) are, in this case, essentially observations of the same variable, the pdf is simplified as
\[ p(y|x_1, x_2) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \exp \left\{ - \frac{\sigma_1^2 (y - \bar{x}_1)^2 - 2\rho \sigma_1 \sigma_2 (y - \bar{x}_1)(y - x_2) + \sigma_2^2 (y - x_2)^2}{2\sigma_1^2 \sigma_2 (1 - \rho^2)} \right\}. \] (16)

In general, the combined variance does not improve if \( \sigma_1 = \rho \sigma_2 \) and/or \( \sigma_2 = \rho \sigma_1 \), as already described in the previous section.

III. APPLICATION TO GRAPH-BASED RECEIVERS

Factor graphs provide a powerful graphical framework that, in conjunction with the sum-product algorithm, has been applied to a large variety of problems in digital communications. The factor graph presented in [5] performs joint channel estimation and data detection for MIMO-OFDM. The factor graph structure, shown in Fig. 1, is based on the following system model:
\[ y[l, k] = H[l, k] x[l, k] + n[l, k], \] (17)
correlation within the channel and/or a loopy graph structure. Correspondingly, the messages \( p(h_1) \) and \( p(h'_1) \) will also be correlated, which has to be considered when \( p(h'_1) \) is combined with \( p(h_3) \) and/or \( p(h_4) \). In general, it is difficult to determine the exact correlation between \( p(h_1) \) and \( p(h'_1) \). However, it is closely related to the correlation between \( p(h_3) \) and \( p(h_2) \). By averaging over the messages \( p(h_1) \) and \( p(h'_1) \) of one OFDM symbol with \( N \) OFDM subcarriers, the correlation between \( p(h_1) \) and \( p(h'_1) \) can be well approximated. Obviously, the accuracy depends on the length of \( N \). Similarly, the correlation coefficient between \( L \)-values can be determined.

IV. NUMERICAL RESULTS

In the following, a MIMO system with two transmit and two receive antennas is assumed. The individual data streams are QPSK modulated and encoded with a rate 1/2 turbo code. A fixed number of 10 iterations is used for the graph-based receiver. The WINNER C2 NLOS channel is applied at a center frequency of \( f_c = 4 \) GHz. The channel matrix generated by this channel model is mutually correlated. \( N = 144 \) OFDM subcarriers and \( K = 14 \) OFDM symbols are chosen. The correlation between messages of different nodes is calculated by averaging over \( N \) OFDM subcarriers for each iteration, as described in the previous section. The complexity of the combining is hereby negligible as already available variance information are combined similarly with the added term of the correlation coefficient. If no correlation is present, the complexity is the same.

The bit error rates (BER) of the original graph-based soft iterative receiver (GSIR) of [5] termed ‘2D-GSIR’ as well as an updated graph-based receiver considering correlation in the message combining step, denoted ‘2D-CGSIR’, are given in Fig. 2 for a velocity of \( v = 60 \) km/h and \( v = 120 \) km/h, respectively. For comparison, a Wiener filter for channel estimation in combination with an APP detector, as well as an APP detector with perfect channel state information are included. The performance improvement due to the correlated combining is 1.2 dB and 2.8 dB for \( v = 60 \) km/h and \( v = 120 \) km/h, respectively. The loss due to channel estimation is about 1.3 dB. As can be seen in Fig. 2a and Fig. 2b, the chosen burst length shows to be sufficiently long for a good approximation of the correlation coefficients. Furthermore, a similar BER performance is obtained with the 2D-CGSIR compared to the WIENER+APP receiver. At the same time, the complexity of the 2D-CGSIR employing Gaussian detection is significantly lower in terms of data detection and channel estimation [5].

V. CONCLUSIONS

In this paper, methods to combine correlated random measures are presented. Two common cases are explained for the message combination. First, the combination of correlated observations for one variable \( p(y|x) \) and, second, the combination of correlated variables for one observation \( p(y|x) \). Additionally, the presented methods are applied to a graph-based iterative receiver. Monte Carlo simulation results are provided which illustrate the significant improvements by taking correlation into account. For future work, the application to factor graphs with intersymbol interference can be investigated as the achievable performance is impaired due to short cycles and the inherent correlation of \( L \)-values.

REFERENCES


