
© 2004 IEEE. Personal use of this material is permitted. However, permission to reprint/republish this material for advertising or promotional purposes or for creating new collective works for resale or redistribution to servers or lists, or to reuse any copyrighted component of this work in other works must be obtained from the IEEE.
Parameter Optimization for a Symmetric Multicarrier Transmission Scheme

Yi Wang
Information and Coding Theory Lab, University of Kiel
Kaisersr. 2, D-24143, Germany. Phone: +49 431 880 6156
E-mail: yw@tf.uni-kiel.de Web: http://www-ict.tf.uni-kiel.de/

ABSTRACT
Transmitter optimization for a multicarrier system with Gaussian pulse as transmit filter is studied in this paper. Symbol duration \( T \) and \( D \) are optimization parameters, where \( D \) is a parameter in Gaussian pulse function. 2-D minimum mean square error decision-feedback equalizer (MMSE-DFE) is used in the receiver, and cost function is the output unbiased signal-to-noise ratio (SNR) of 2-D MMSE-DFE. Four kinds of deterministic channels are discussed. Analysis shows that the problem for time-selective channel is the dual problem for frequency-selective channel. If doubly-selective channel is deterministic separable, we obtain a general result, which is highly symmetric in time and frequency domains. Finally, we propose balanced design for practical purposes.

II. SYSTEM MODEL AND PROBLEM FORMULATION

2.1 The multicarrier Transmission Scheme

The Gaussian pulse and its Fourier transform are given by

\[
\begin{align*}
g(t) &= \frac{\sqrt{2}}{\sqrt{\alpha T}} \exp\left( -\pi \frac{t^2}{\alpha T} \right), \\
G(f) &= \frac{\sqrt{2}}{\sqrt{\alpha T}} \exp\left( -\pi \alpha T f^2 \right),
\end{align*}
\]

where \( \alpha \) is a parameter for the adjustment of the pulse. When \( \alpha = 1 \), the Gaussian pulse has exactly the same shape in time and frequency domain. MMSE-DFE is employed in the receiver. Analytical results will be given for different fading channels.

Fig. 1 The multicarrier transmission scheme

The multicarrier transmission scheme in study is depicted in Fig. 1, where the notations are interpreted as follows:

- \( m, n \): carrier and symbol index, respectively
- \( \{a(m,n)\} \): complex data symbols with average symbol energy \( E \)
- \( M \): number of carriers
- \( N \): number of data symbols per carrier, or block length
- \( T \): symbol duration
- \( \Delta f \): carrier spacing
- \( g(t) \): Gaussian pulse

The Gaussian pulse and its Fourier transform are given by

\[
g(t) = \frac{\sqrt{2}}{\sqrt{\alpha T}} \exp\left( -\pi \frac{t^2}{\alpha T} \right), \\
G(f) = \frac{\sqrt{2}}{\sqrt{\alpha T}} \exp\left( -\pi \alpha T f^2 \right),
\]

where \( \alpha \) is a parameter for the adjustment of the pulse. When \( \alpha = 1 \), the Gaussian pulse has exactly the same shape in time and frequency domain. Next, we give two constraints on the transmitter:

(i) \( \Delta T = 1 \), which means that the system achieves the maximum spectral efficiency;
(ii) \( M >> 1 \) and \( N >> 1 \), in order to use 2-D MMSE-DFSE in the receiver.

According to Fig. 1, the transmitted signal and its Fourier transform can be expressed as

\[
x(t) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} a(m,n)g(t - nT)e^{j2\pi mn\Delta f}.
\]
If we do such exchanges, \( g(t) \leftrightarrow G(f) \), \( t \leftrightarrow f \), \( n \leftrightarrow m \), \( N \leftrightarrow M \), \( T \leftrightarrow \Delta f \), then the transmitted signal also has the same shape in time and frequency domain, i.e., it is highly symmetric in time-frequency plane.

### 2.3 2-D Formal Discrete-Time Signal Model

In a general communication system, modulator, physical channel and demodulator can be modeled as the equivalent discrete-time (EDT) channel [5][8]. For the multiscarrier system, the EDT channel can be modeled as a 2-D linear filter, and the output samples of the demodulator can be expressed as

\[
y(m,n) = \sum_{m' \in \mathbb{Z}} \sum_{n' \in \mathbb{Z}} a(m',n') h(m-m',n-n') + v(m,n),
\]

where \( h(m,n) \) is the 2-D EDT channel coefficients, \( v(m,n) \) is the 2-D noise, and \( * \) represents 2-D convolution. Moreover, model (4) is called formal if the autocorrelation of the noise is proportional to the channel impulse response, i.e.,

\[
E[v(m,n)v^*(m',n')] = N_0 h(m-m',n-n'),
\]

where \( N_0 \) is the power spectral density of white Gaussian noise from the physical channel. The formal model arises from the system where matched filter is used before equalization [5][6].

When the EDT channel is 2-D variant, the formal signal model can be written as

\[
y(m,n) = \sum_{m' \in \mathbb{Z}} \sum_{n' \in \mathbb{Z}} a(m',n') h(m-m';n,n-n') + v(m,n),
\]

\[
E[v(m,n)v^*(m',n')] = N_0 h(m,m-m';n,n-n').
\]

### 2.4 Problem Formulation

If the input signal of the 2-D MMSE-DFE (with infinite length) can be written in the formal signal model (4)(5), and there are no error propagations in feedback, the output unbiased SNR can be expressed as [5]

\[
SNR_{\text{MMSE-DFE}} = \exp \left\{ \frac{1}{4\pi^2} \int_{-T}^{T} \int_{-W}^{W} \log \left[ 1 + \frac{E[H(w_1,w_2)]}{N_0} \right] dw_1 dw_2 \right\} - 1,
\]

where

\[
H(w_1,w_2) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h(m,n)e^{-j2\pi mw_1} e^{-j2\pi nw_2},
\]

is the 2-D Fourier transform of the EDT channel. Consequently, the optimization problem can be formulated as

\[
[\alpha, T] = \arg \max_{\alpha, T} SNR_{\text{MMSE-DFE}}(\alpha, T).
\]

It is shown from (7) that the cost function \( SNR_{\text{MMSE-DFE}} \) depends on \( h(m,n) \) and \( E_\alpha / N_0 \), while \( h(m,n) \) depends only on the Gaussian pulse and the physical channel. Moreover, the cost function has a simple relation to the channel capacity [5], and thus maximizing (7) is equivalent to maximizing the capacity.

### III. OPTIMIZATION FOR DIFFERENT CHANNELS

According to (9) and (7), the key issue for the optimization problem is to calculate the 2-D EDT channel. In the next derivations, the following input-output relation of a time-varying linear filter will be frequency used:

\[
y(t) = x(t) * h(t,\tau) = \int_{-\infty}^{\infty} x(t-\tau) h(t,\tau) d\tau.
\]

For ease of description, we are not ready to present the derivation details.

### 3.1 AWGN Channel

In AWGN channel, the physical channel is \( c(t,\tau) = \delta(t) \) and hence the FMCIR, in terms of (3), is \( c_{\text{FMCIR}}(t,\tau) = \delta(t) \). Based on the system model and using (10), we can obtain that the 2-D EDT channel is

\[
h(m,n) = h_0(m,n) := \int g(\tau + nT)g^*(\tau)e^{-j2\pi mw_1} e^{-j2\pi nw_2} d\tau = e^{-\left(\frac{2\pi}{w_1}\right)^2} e^{-\left(\frac{2\pi}{w_2}\right)^2},
\]

and the noise satisfies (5). Thus, the system is equivalent to the 2-D formal signal model.

- If \( \Delta fT \) is not equal to an integer, the EDT channel is variant in the two dimensions, but (6b) is satisfied.
- The EDT channel is symmetric on \( m \) and \( n \), i.e., \( h_0(-m,-n) = h_0(m,n) \). Moreover, when \( \alpha = 1 \), \( m \) and \( n \) are exchangeable, i.e., \( h_0(m,n) = h_0(n,m) \).
- The EDT channel depends only on \( \alpha \), and

\[
SNR_{\text{MMSE-DFE}}(\alpha) = SNR_{\text{MMSE-DFE}}(1/\alpha).
\]

The cost function (7) versus \( \alpha \) and \( E_\alpha / N_0 \) is plotted in Fig. 3. We can see that \( \alpha = 1 \) is optimal for different \( E_\alpha / N_0 \). Moreover, it appears that \( E_\alpha / N_0 \) does not change the shape of the cost function.
function. For this reason, we assume that $E_s/N_0 = 20dB$ in the following examples.

![Fig.3 System performance in AWGN channel](image)

### 3.2 Frequency-Selective Channel

In frequency-selective channel, $c(t,\tau) = c(\tau)$, and the FMCIR is $c_{FMCIR}(t,\tau) = c^\ast (-\tau)$ . The EDT channel is then

$$h(m, m - m'; n, n - n') = \int R_{cc}(\tau)e^{-j2\pi m'n'/T}h_0(m - m', n - n')d\tau,$$

(12)

where

$$R_{cc}(\tau) := \int c(t)e^{-j\tau t}dt = c(t)\ast c^\ast (-\tau)|_{-\infty},$$

(13)

and the noise satisfies (6b). Thus, the system is equivalent to the extended 2-D formal signal model (6a)(6b).

- The EDT channel depends on $\alpha$ and $T$;
- $h(m, m - m'; n, n - n') = h(m, m - m'; n - n')$, i.e., the EDT channel is variant with $m$ and invariant with $n$. The reason is that the frequency-selective channel causes different fading on each carrier.

For extended 2-D FDT signal model, one method is to do optimization for each carrier and symbol, and then average the cost function over all carriers and symbols. However, a channel with finite length in time has limited energy, and its magnitude of frequency response degrades to zero as the frequency increases. Thus, the method that doing average over all carriers and symbols will make the cost function to zero. For simplicity, we assume that $m = 0$ in the following example.

**Example 1**: Consider a physical channel with impulse response

$$c(\tau) = \begin{cases} \frac{1}{\tau_{\max}} & |\tau| \leq \tau_{\max}/2, \text{ and } \tau_{\max} = 0.1s. \\ 0 & \text{else} \end{cases}$$

![Fig. 4 Numerical result of example 1](image)

Numerical results are plotted in Fig.4. It is shown that $SNR_{MMSE-DFE,U}$ increases with symbol duration. Thus, $T \to \infty$ is suggested. In this case, delay spread is negligible compared with symbol duration, and the EDT channel approaches to the one in AWGN channel. This result can be seen from (13) since

$$\lim_{T \to \infty} \text{exp}(-j2\pi m'n'/T) = 1 \text{ and } \lim_{T \to \infty} R_{cc}(\tau) = \delta(\tau).$$

It is also shown from Fig. 4 that $\alpha = 1$ is optimal. But if symbol duration is not well designed, $\alpha = 1$ is not optimal, e.g. $\alpha > 1$ is better when $T < 0.1$.

### 3.3 Time-Selective Channel

In time-selective channel, $c(t,\tau) = c(t)\delta(\tau)$ . Since perfect channel information is known, we can assume that $c(t)$ is a deterministic signal, i.e. $C(f) = \int c(t)e^{-j2\pi ft}dt$ is a known function. The FMCIR is then $c_{FMCIR}(t,\tau) = c^\ast(t)\delta(\tau)$ . In terms of time-frequency duality, as discussed in [9], communication in time-selective channel is the dual problem of communication in frequency-selective channel. Therefore, the solution for time-selective channel can be found directly from the solution of its duals by interchanges of dual roles [9].

Based on this conclusion, the 2-D EDT channel for time-selective channel is

$$h(m, m - m'; n, n - n') = \int R_{cc}(\nu)e^{j2\pi m'n'/T}h_0(m - m', n - n')d\nu,$$

(14)

where

$$R_{cc}(\nu) := \int C(\nu)C^\ast(v-f)dv = C(f)\ast C^\ast(-f),$$

and the system is also equivalent to the extended 2-D formal signal model.

From (14), we see that the EDT channel is invariant with $m$ but variant with $n$. Similarly, this is because that time-selective channel produces different fading on each carrier. For simplicity, we assume that $n = 0$ in the next example.

**Example 2**: Consider a channel with frequency impulse response

$$C(f) = \begin{cases} \frac{1}{2f_{D\max}} & |f| \leq f_{D\max}, \text{ and } f_{D\max} = 5Hz. \\ 0 & \text{else} \end{cases}$$

![Fig. 5 Numerical result of example 2](image)

Numerical results are plotted in Fig. 5. It is similar with Fig. 4 except $\Delta f$ is used instead of $T$ . Thus, $\alpha = 1$ and $\Delta f \to \infty$, or $T \to 0$ is optimal for time-selective channel.

### 3.4 Doubly-Selective Channel

For a linear doubly-selective channel with impulse response $c(t,\tau)$, the corresponding FMCIR is $c_{FMCIR}(t,\tau) = c^\ast(t,-\tau)$ . Based
on the system model and using (10) to calculate the signal model, 
we found that it is impossible to get a 2-D formal signal model as 
in (6), and hence 2-D MMSE-DFE does not work. Thus, we make 
some assumptions on c(t, r) so that the optimization problem can be approached.

A channel is separable if 
\[ c(t, r) = c_1(t)c_2(r), \]  
(15)
The channel is deterministic separable if \( c_1(t) \) is deterministic, i.e., its Fourier transform is a known function. In terms of (15), AWGN channel, frequency-selective channel and time-selective channel discussed in the above are all special cases of separable deterministic channels. Consequently, the FMCIR of a separable deterministic channel is \( c_{\text{FMCIR}}(t, r) = c_1(t)c_2(-r) \), and the EDT channel is 
\[ h(m, m - m'; n, n - n') = \left\{ \begin{array}{ll} \mathbb{I} & \text{if } \frac{f}{\Delta f} < f_{\text{Dmax}}, f_{\text{Dmax}} < \tau_{\text{max}} = 0.1s \\
0 & \text{else} \end{array} \right. \]  
(16)
where \( R_{cc}^{(2)}(\tau) = \mathbb{I}c_1(t)c_1(t - \tau) d\tau \), \( R_{cc}^{(1)}(f) = \int C(v)C^*(v - f) dv \), and \( C(f) \) is the Fourier transform of \( c_1(t) \).

- Let \( R_{cc}^{(1)}(f) = \delta(f) \), we can get the result for frequency-selective channel, as in (12). Let \( R_{cc}^{(2)}(\tau) = \delta(\tau) \), we can get the result for time-selective channel, as in (14).
- Both \( f \) (frequency) and \( \tau \) (time) play the same roles in (16), which is mainly due to the symmetric system model and the time-frequency duality.
- In (16), all required information to describe the impact of the fading channel on the EDT channel are the autocorrelations of the physical channel, \( R_{cc}^{(2)}(\tau) \) and \( R_{cc}^{(1)}(f) \).
- \( h(m, m - m'; n, n - n') \) is variant with both \( m \) and \( n \), because doubly-selective channel causes different fading on each carrier and symbol. For simplicity, we care about the optimization problem at \( n = 0 \) and \( m = 0 \).

Example 3: Consider a deterministic separable channel with impulse response \( f_{\text{Dmax}} = 5 \text{Hz, } \tau_{\text{max}} = 0.1s \)

\[ c_1(f) = \left\{ \begin{array}{ll} \frac{1}{2f_{\text{Dmax}}} & |f| < f_{\text{Dmax}} \\
0 & \text{else} \end{array} \right. \]  
(17)
\[ c_2(\tau) = \left\{ \begin{array}{ll} \frac{1}{\tau_{\text{max}}} & |\tau| < \frac{\tau_{\text{max}}}{2} \\
0 & \text{else} \end{array} \right. \]  
(17)
that the optimal point is \( T = \sqrt{f_{\text{Dmax}}/2f_{\text{Dmax}}} \) and \( \alpha = 1 \), which satisfies
\[ \alpha = 1/\alpha, \]  
(18a)
\[ \tau_{\text{max}}/T = 2f_{\text{Dmax}}/\Delta f. \]  
(18b)
Eq. (18a) means that Gaussian pulse has exactly the same shape in time and frequency domain (cf. (1)). Eq. (18b) means that the EDT channel has the same channel memory length in time and frequency domain. In sense of that, we call (18) as balanced design.

Balanced design is the optimal solution for the multicarrier system over a doubly-selective channel given by (16), where delay power profile and Doppler power profile are both rectangular waveforms. In practice, the physical channel may have much different power profiles in delay and Doppler directions. Then balanced design is possibly not the optimal solution. But it is much helpful for a design without any information about the designed channel in practical engineering.

**ACKNOWLEDGEMENT**
The author wishes to thank Prof. Peter A. Hoehler for his ideas on the balanced design and suggestions on clarifying the meaning. I am grateful to Xiao-Ming Chen for his discussion on channel normalization.

**REFERENCES**

---

**Fig. 6 Numerical result of example 3**

Numerical results are plotted in Fig. 6. It is shown that the optimal point for \( \text{SNR}_{\text{MMSE-DFE,U}} \) is \( T = 0.1 \text{s} \) and \( \alpha = 1 \). In fact, by changing the parameters \( f_{\text{Dmax}} \) and \( \tau_{\text{max}} \) in (17), the results show...